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The Schwinger Formula Revisited II (A Mathematical Treatment)

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We study the production of pairs in non-analytic potentials. While, when the potential is analytic, the average number of produced pairs is exponentially small in $\hbar$, when the potential is non-analytic, using the W.K.B. method, we prove that the average number of produced pairs is $\mathcal{O}(\alpha\hbar^{2N})$, where N is the regularity of the potential and α is the fine structure constant. We give a rigorous proof of the Schwinger's formula.

KEY WORDS: Pair production; Schwinger's formula; W.K.B. method.

1. INTRODUCTION

In the previous paper (Haro, 2003) we have obtained the average number of produced pairs due to the presence of an external uniform field. The most important application of this formula is the "formal" deduction of the average number of produced pairs in a constant electric field.

Here, the objective is the study of the average number of produced pairs in non-analytic fields (the case of an analytic potential has been studied in Eisenberg and Kälberman (1988), Marinov and Popov (1977), Popov (1972) and the rigorous computation of the average number of produced pairs in a constant electric field.

First, we will see the relation that exists between the probability that a pair is created and the transmission and reflection coefficients of the associated Klein–Gordon equation. It is well-known fact (Berry, 1982; Fulling, 1985) that these coefficients depend on the regularity of the field, thus, the average number of produced pairs depends on the regularity of the field. This fact is explained in detail in Section 2.

In Section 3, we give bounds of the average number of produced pairs in a uniform electric field of regularity $\mathcal{C}^{N-1}$. For this we use the W.K.B. method in a similar way to Berry (1982), and we give bounds of the error obtained using this

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method. We prove that the average number of produced pairs is $\mathcal{O}(\alpha\hbar^{2N})$, where α is the fine structure constant. And we see that, in the case that the electric field has a discontinuity at the point T, the average number of produced pairs, when the field is switched off, is in the semi-classical approximation

$$\frac{\alpha E(T)}{64mc^2}$$

where $E(T)$ is the energy of the field at time T.

Finally in Section 4, we prove that the average number of produced pairs per unit time and unit volume in a constant electric field, when the time increases to infinity, is

$$\frac{E^2\alpha}{8\pi^3\hbar}\exp\left(-\frac{\pi m^2c^4}{\hbar ceE}\right)$$

This result has been formally proved for some authors (Haro, 2003; Holstein, 1999; Nikishov, 1970). In this work we give a very large and complicated demonstration, because we use the asymptotic expressions of the confluent hypergeometric functions with the bounds of the remaining terms, and also, it is used a semiclassical solution of the second quantized Klein–Gordon field equation, in the Schödinger picture. But anyway, we believe that a rigorous demonstration and a suitable interpretation of this result was needed.

2. PAIR PRODUCTION PROCESS

As in the previous paper, we consider the Klein–Gordon field in a box of volume L^3, coupled with an external uniform vector potential $\vec{f}(t)$. The Hamilton equations are

$$\ddot{u}_{\vec{k}} + \omega^2_{\vec{k}}(t)u_{\vec{k}} = 0, \quad \vec{k} \in \mathbb{Z}^3 \tag{1}$$

with

$$\omega^2_{\vec{k}}(t) = \frac{1}{\hbar^2}\left(c^2\left|\frac{2\pi\hbar\vec{k}}{L} + \frac{e}{c}\vec{f}(t)\right|^2 + m^2c^4\right) \tag{2}$$

We suppose that $\omega_{\vec{k}}(t)$ has the following form

$$\omega_{\vec{k}}(t) = \begin{cases} \omega_{1,\vec{k}} & \text{if } t < T_1 \\ \omega_{\vec{k}}(t) \in \mathcal{C}^\infty(T_1, T_2) & \text{if } T_1 < t < T_2 \\ \omega_{2,\vec{k}} & \text{if } t > T_2 \end{cases} \tag{3}$$

We look for solutions of (1) in the following form

$$u_{\vec{k}}(t) = \begin{cases} \phi^{+}_{1,\vec{k}}(t) & \text{if } t < T_1 \\ A_{\vec{k}}\varphi^{+}_{\vec{k}}(t) + B_{\vec{k}}\varphi^{-}_{\vec{k}}(t) & \text{if } T_1 < t < T_2 \\ a_{\vec{k}}\phi^{+}_{2,\vec{k}}(t) + b_{\vec{k}}\varphi^{-}_{2,\vec{k}}(t) & \text{if } t > T_2 \end{cases} \tag{4}$$

where

$$\phi^{\pm}_{\binom{1}{2}}(t) = \frac{\exp\Big(\pm i\omega_{\binom{1}{2},\vec{k}}\Big(t - T_{\binom{1}{2}}\Big)\Big)}{\sqrt{2\omega_{\binom{1}{2},\vec{k}}\hbar}} \tag{5}$$

Therefore, the average number of produced pairs in the $\vec{k}$-state at time $t > T_2$ is (Fulling, 1985)

$$N_{\vec{k}}(t > T_2) = |b_{\vec{k}}|^2 \tag{6}$$

and the total average number of produced pairs at time $t > T_2$, is

$$N(t > T_2) = \sum_{\vec{k}\in\mathbb{Z}^3} |b_{\vec{k}}|^2 \tag{7}$$

In order to calculat $a_{\vec{k}}$ and $b_{\vec{k}}$, we assume that $u_{\vec{k}}(t) \in \mathcal{C}^1(\mathbb{R})$, then we obtain

$$\begin{aligned} a_{\vec{k}} = {} & \frac{i\hbar}{W[\varphi^{+}_{\vec{k}}(T_1^{+}), \varphi^{-}_{\vec{k}}(T_1^{+})]}[W[\phi^{+}_{1,\vec{k}}(T_1^{-}), \varphi^{-}_{\vec{k}}(T_1^{+})]W[\varphi^{+}_{\vec{k}}(T_2^{-}), \phi^{-}_{2,\vec{k}}(T_2^{+})] \\ & -W[\phi^{+}_{1,\vec{k}}(T_1^{-}), \varphi^{+}_{\vec{k}}(T_1^{+})]W[\varphi^{-}_{\vec{k}}(T_2^{-}), \phi^{-}_{2,\vec{k}}(T_2^{+})]] \end{aligned} \tag{8}$$

$$\begin{aligned} b_{\vec{k}} = {} & \frac{-i\hbar}{W[\varphi^{+}_{\vec{k}}(T_1^{+}), \varphi^{-}_{\vec{k}}(T_1^{+})]}\big[W[\phi^{+}_{1,\vec{k}}(T_1^{-}), \varphi^{-}_{\vec{k}}(T_1^{+})]W[\varphi^{+}_{\vec{k}}(T_2^{-}), \phi^{+}_{2,\vec{k}}(T_2^{+})] \\ & -W[\phi^{+}_{1,\vec{k}}(T_1^{-}), \varphi^{+}_{\vec{k}}(T_1^{+})]W[\varphi^{-}_{\vec{k}}(T_2^{-}), \phi^{+}_{2,\vec{k}}(T_2^{+})]\big], \end{aligned} \tag{9}$$

where $W[f(t), g(t)]$ is the Wronskian of the functions f and g at the point t, and

$$f(T^{+}) = \lim_{\substack{t\to T\\ t>T}} f(t); \quad f(T^{-}) = \lim_{\substack{t\to T\\ t>T}} f(t)$$

Remark 2.1. As a consequence of the constancy of the Wronskian, we have $|a_{\vec{k}}|^2 - |b_{\vec{k}}|^2 = 1$.

Remark 2.2. If we take $\varphi^{\pm}_{\vec{k}}(t)$ such that

$$\begin{cases} \phi^{\pm}_{1,\vec{k}}(T_1^{-}) = \varphi^{\pm}_{\vec{k}}(T_1^{+}) \\ \dot{\phi}^{\pm}_{1,\vec{k}}(T_1^{-}) = \dot{\varphi}^{\pm}_{\vec{k}}(T_1^{+}). \end{cases} \tag{10}$$

Then, we obtain the following expression

$$|b_{\vec{k}}|^2 = \hbar^2 \left| W[\varphi_{\vec{k}}^{+}(T_2), \phi_{2,\vec{k}}^{+}(T_2)] \right|^2. \tag{11}$$

For this reason, we will always demand that the condition (10) is satisfied.

Now, in order to calculate $\varphi_{\vec{k}}^{\pm}(t)$, we use the WKB method. We write (Berry, 1982)

$$\varphi_{\vec{k}}^{\pm}(t) = \frac{1}{\sqrt{2\epsilon_{\vec{k}}(T_1)}} \exp\left(\pm \frac{i}{\hbar} \int_{T_1}^{t} P_{\vec{k}}^{\pm}(\tau) d\tau \right) \tag{12}$$

After the substitution in (1), we obtain $\pm i\hbar \dot{P}_{\vec{k}}^{\pm}(t) = (P_{\vec{k}}^{\pm}(t))^2 - \epsilon_{\vec{k}}^2(t)$.

We expand $P_{\vec{k}}^{\pm}$ in power series of $\hbar$ thus, $P_{\vec{k}}^{\pm}(t) = \sum_{n=0}^{\infty} \hbar^n P_{n,\vec{k}}^{\pm}(t)$. We obtain, after having equalized the powers of $\hbar$

$$P_{0,\vec{k}}^{\pm}(t) = \epsilon_{\vec{k}}(t); \quad P_{1,\vec{k}}^{\pm}(t) = \pm \frac{i\dot{\epsilon}_{\vec{k}}(t)}{2\epsilon_{\vec{k}}(t)}; \tag{13}$$

Now, let $\varphi_{\mathrm{WKB},\vec{k}}^{n,\pm}(t)$ be the semi-classical approximation of order n of $\varphi_{\vec{k}}^{\pm}(t)$, then

$$\varphi_{\mathrm{WKB},\vec{k}}^{1,\pm}(t) = \frac{1}{\sqrt{2\epsilon_{\vec{k}}(t)}} \exp\left(\pm \frac{i}{\hbar} \int_{T_1}^{t} \epsilon_{\vec{k}}(\tau) d\tau \right); \tag{14}$$

2.1. W.K.B. Bounds

Here, we prove a theorem that gives the bound of the error obtained using the W.K.B. method.

Theorem 2.1. *If we assume that $\omega_{\vec{k}}(t)$ is monotonic in the interval (T_1, T_2), then, for the differential equation*

$$\ddot{u}_{\vec{k}} + \omega_{\vec{k}}^2(t) u_{\vec{k}} = g_{\vec{k}}(t) \tag{15}$$

we have the following bounds

(A) When $\dot{\omega}_{\vec{k}}(t) \geq 0$

$$|u_{\vec{k}}(t)| \leq \frac{\hbar}{\epsilon_{\vec{k}}(T_1)} \sqrt{|\dot{u}_{\vec{k}}(T_1)|^2 + \frac{\epsilon_{\vec{k}}^2(T_1)}{\hbar^2} |u_{\vec{k}}(T_1)|^2} + \hbar \int_{T_1}^{t} \frac{|g_{\vec{k}}(\tau)|}{\epsilon_{\vec{k}}(\tau)} d\tau \tag{16}$$

(B) When $\dot{\omega}_{\vec{k}}(t) \leq 0$

$$|u_{\vec{k}}(t)| \leq \frac{\hbar}{\epsilon_{\vec{k}}(t)} \left[\sqrt{|\dot{u}_{\vec{k}}(T_1)|^2 + \frac{\epsilon_{\vec{k}}^2(T_1)}{\hbar^2} |u_{\vec{k}}(T_1)|^2} + \hbar \int_{T_1}^{t} |g_{\vec{k}}(\tau)| d\tau \right]. \tag{17}$$

Proof: (A) From the equation (15), we obtain

$$\frac{1}{\omega_{\vec{k}}^2(t)}\frac{d|\dot{u}_{\vec{k}}|^2}{dt}+\frac{d|u_{\vec{k}}|^2}{dt}=\frac{1}{\omega_{\vec{k}}^2(t)}(g_{\vec{k}}(t)\dot{u}_{\vec{k}}^*+g_{\vec{k}}^*(t)\dot{u}_{\vec{k}}). \tag{18}$$

Consequently

$$\frac{d}{dt}\left(\frac{|\dot{u}_{\vec{k}}|^2}{\omega_{\vec{k}}^2(t)}+|u_{\vec{k}}|^2\right)=-\frac{2\dot{\omega}_{\vec{k}}(t)}{\omega_{\vec{k}}^3(t)}|\dot{u}_{\vec{k}}|^2+\frac{1}{\omega_{\vec{k}}^2(t)}(g_{\vec{k}}(t)\dot{u}_{\vec{k}}^*+g_{\vec{k}}^*(t)\dot{u}_{\vec{k}})\leq\frac{2|g_{\vec{k}}(t)||\dot{u}_{\vec{k}}|}{\omega_{\vec{k}}^2(t)}. \tag{19}$$

From this result we can deduce

$$\frac{d}{dt}\sqrt{\frac{|\dot{u}_{\vec{k}}|^2}{\omega_{\vec{k}}^2(t)}+|u_{\vec{k}}|^2}\leq\frac{|g_{\vec{k}}(t)|}{\omega_{\vec{k}}(t)} \tag{20}$$

Now, after integrating we obtain the result.

(B) From Eq. (16), we have

$$\frac{d|\dot{u}_{\vec{k}}|^2}{dt}+\omega_{\vec{k}}^2(t)\frac{d|u_{\vec{k}}|^2}{dt}=g_{\vec{k}}(t)\dot{u}_{\vec{k}}^*+g_{\vec{k}}^*(t)\dot{u}_{\vec{k}}. \tag{21}$$

Consequently,

$$\frac{d}{dt}\left(|\dot{u}_{\vec{k}}|^2+\omega_{\vec{k}}^2(t)|u_{\vec{k}}|^2\right)=2\dot{\omega}_{\vec{k}}(t)\omega_{\vec{k}}(t)|u_{\vec{k}}|^2+g_{\vec{k}}(t)\dot{u}_{\vec{k}}^*+g_{\vec{k}}^*(t)\dot{u}_{\vec{k}}\leq 2|g_{\vec{k}}(t)||u_{\vec{k}}|, \tag{22}$$

and we have

$$\frac{d}{dt}\sqrt{|\dot{u}_{\vec{k}}|^2+\omega_{\vec{k}}^2(t)|u_{\vec{k}}|^2}\leq|g_{\vec{k}}(t)| \tag{23}$$

and the result is obtained after integration. □

Corollary 2.1. *In the case that $\dot{\omega}_{\vec{k}}$ has a finite number of zeros in (T_1, T_2). If we assume that $u_{\vec{k}}(T_1)=\dot{u}_{\vec{k}}(T_1)=0$, and we suppose that $\omega_{\vec{k}}$ is a bounded function in $\mathbb{R}$ then, it exist an a-dimensional constant C independent of $\hbar, T_1, T_2$ and $\vec{k}$, such that*

$$|u_{\vec{k}}|\leq\hbar\frac{C}{\epsilon_{\vec{k}}}\|g_{\vec{k}}\|_1;\quad|\dot{u}_{\vec{k}}|\leq C\|g_{\vec{k}}\|_1, \tag{24}$$

where $\epsilon_{\vec{k}}=\sqrt{\frac{4\pi^2c^2\hbar^2|\vec{k}|^2}{L^2}+m^2c^4}$ and $\|g_{\vec{k}}\|_1=\int_{T_1}^{T_2}|g_{\vec{k}}(\tau)|d\tau$.

Remark 2.3. The bound is also valid when $|T_2-T_1|=+\infty$.

3. APPLICATION OF THE W.K.B. METHOD

Let $\varphi_{\vec{k}}^{\pm}$ be the solutions of (1) in the interval (T_1, T_2), that verify

$$\begin{cases} \varphi_{\mathrm{WKB},\vec{k}}^{n,\pm}(T_1) = \varphi_{\vec{k}}^{\pm}(T_1) \\ \dot{\varphi}_{\mathrm{WKB},\vec{k}}^{n,\pm}(T_1) = \dot{\varphi}_{\vec{k}}^{\pm}(T_1) \end{cases} \tag{25}$$

Due to Corollary 2.1 we have the following bounds

$$\left|\varphi_{\vec{k}}^{\pm}(t) - \varphi_{\mathrm{WKB},\vec{k}}^{n,\pm}(t)\right| \leq \frac{\hbar C}{\epsilon_{\vec{k}}} \left\| \left[\pm \frac{i}{\hbar} \dot{\tilde{P}}_{n,\vec{k}}^{\pm} - \frac{1}{\hbar^2}\left((\tilde{P}_{n,\vec{k}}^{\pm})^2 - \epsilon_{\vec{k}}^2\right) \right] \varphi_{\mathrm{WKB},\vec{k}}^{n,\pm} \right\|_1 \tag{26}$$

$$\left|\dot{\varphi}_{\vec{k}}^{\pm}(t) - \dot{\varphi}_{\mathrm{WKB},\vec{k}}^{n,\pm}(t)\right| \leq C \left\| \left[\pm \frac{i}{\hbar} \dot{\tilde{P}}_{n,\vec{k}}^{\pm} - \frac{1}{\hbar^2}\left((\tilde{P}_{n,\vec{k}}^{\pm})^2 - \epsilon_{\vec{k}}^2\right) \right] \varphi_{\mathrm{WKB},\vec{k}}^{n,\pm} \right\|_1 \tag{27}$$

where $\tilde{P}_{n,\vec{k}}^{\pm} = \sum_{j=0}^{n} \hbar^j P_{j,\vec{k}}^{\pm}$.

Therefore, if we define

$$G_n\left(\frac{e}{mc^2}\right) = \frac{1}{n+1} \sum_{\substack{j_1,\ldots,j_{n+1}=0 \\ i_1,\ldots,i_{n+1}=1 \\ \sum_{k=1}^{n+1} i_k j_k = n+1}} \left\| \prod_{k=1}^{n+1} (D^{i_k} \vec{f})^{j_k} \right\|_1 \left(\frac{e}{mc^2}\right)^{\sum_{k=1}^{n+1} j_k} \tag{28}$$

we obtain

$$|\varphi_{\vec{k}}^{\pm}(t) - \varphi_{\mathrm{WKB},\vec{k}}^{n,\pm}(t)| \leq \frac{\hbar^n \tilde{C}}{(mc^2)^{n-2} \epsilon_{\vec{k}}^{\frac{5}{2}}} G_n\left(\frac{e}{mc^2}\right) \tag{29}$$

$$|\dot{\varphi}_{\vec{k}}^{\pm}(t) - \dot{\varphi}_{\mathrm{WKB},\vec{k}}^{n,\pm}(t)| \leq \frac{\hbar^{n-1} \tilde{C}}{(mc^2)^{n-2} \epsilon_{\vec{k}}^{\frac{3}{2}}} G_n\left(\frac{e}{mc^2}\right) \tag{30}$$

where $\tilde{C}$ is an a-dimensional constant independent of $\hbar$ and $\vec{k}$.

With these results, we can calculate $|b_{\vec{k}}|^2$. In fact, using the W.K.B. method at the order 2, we have

$$\tilde{P}_{2,\vec{k}}^{\pm}(t) = \epsilon_{\vec{k}}(t) \pm \frac{i\hbar\dot{\epsilon}_{\vec{k}}(t)}{2\epsilon_{\vec{k}}(t)} + \frac{\hbar^2}{2\epsilon_{\vec{k}}(t)}\left[-\frac{1}{2}\frac{d}{dt}\left(\frac{\dot{\epsilon}_{\vec{k}}(t)}{2\epsilon_{\vec{k}}(t)}\right) + \frac{\dot{\epsilon}_{\vec{k}}^2(t)}{4\epsilon_{\vec{k}}^2(t)} \right] \tag{31}$$

We suppose that $\omega_{\vec{k}} \in \mathcal{C}^2$ at the point T_1. In this case

$$\varphi_{\mathrm{WKB},\vec{k}}^{2,\pm}(T_1) = \phi_{\vec{k}}^{\pm}(T_1); \quad \dot{\varphi}_{\mathrm{WKB},\vec{k}}^{2,\pm}(T_1) = \dot{\phi}_{\vec{k}}^{\pm}(T_1) \tag{32}$$

Therefore, if we use Corollary (2.1), the solution of (1) in (T_1, T_2) verifies

$$\varphi_{\vec{k}}^{\pm}(t) = \varphi_{\mathrm{WKB},\vec{k}}^{2,\pm}(t) + \hbar^2 A_{\vec{k}}^{\pm}(t); \quad \dot{\varphi}_{\vec{k}}^{\pm}(t) = \dot{\varphi}_{\mathrm{WKB},\vec{k}}^{2,\pm}(t) + \hbar B_{\vec{k}}^{\pm}(t) \tag{33}$$

with

$$\|A_{\vec{k}}^{\pm}\|_{\infty} \le \frac{\tilde{C}}{\epsilon_{\vec{k}}^{\frac{5}{2}}} G_2\left(\frac{e}{mc^2}\right); \quad \|B_{\vec{k}}^{\pm}\|_{\infty} \le \frac{\tilde{C}}{\epsilon_{\vec{k}}^{\frac{3}{2}}} G^2\left(\frac{e}{mc^2}\right) \tag{34}$$

where $\|g\|_{\infty} = \max_{t\in(T_1,T_2)} |g(t)|$.

Since $\varphi_{\vec{k}}^{\pm}$ verifies the condition (10), if we apply the formula (11), we obtain

$$|b_{\vec{k}}|^2 = \frac{\hbar^2 \dot{\epsilon}_{\vec{k}}^2(T_2)}{16\epsilon_{\vec{k}}^4(T_2)} + \hbar^3 F_1 + \hbar^4 F_2 \tag{35}$$

with

$$|F_1| \le \frac{K}{\epsilon_{\vec{k}}^4} \|e\dot{\vec{f}}\|_{\infty} \left[G_2\left(\frac{e}{mc^2}\right) + \frac{\|e\ddot{\vec{f}}\|_{\infty}}{mc^2} + \frac{\|e\dot{\vec{f}}\|_{\infty}^2}{(mc^2)^2} \right] \tag{36}$$

$$|F_2| \le \frac{K}{\epsilon_{\vec{k}}^4} \left[G_2\left(\frac{e}{mc^2}\right) + \frac{\|e\ddot{\vec{f}}\|_{\infty}}{mc^2} + \frac{\|e\dot{\vec{f}}\|_{\infty}^2}{(mc^2)^2} \right]^2 \tag{37}$$

where K is an a-dimensional constant.

Consequently, in the semi-classical approximation, in the case that $\omega_{\vec{k}} \notin \mathcal{C}^1$ at the point T_2, we have

$$|b_{\vec{k}}|^2 \sim \frac{\hbar^2 \dot{\epsilon}_{\vec{k}}^2(T_2)}{16\epsilon_{\vec{k}}^4(T_2)} \tag{38}$$

Remark 3.1. This result is also valid if we only suppose that $\omega_{\vec{k}} \in \mathcal{C}^1$ at the point T_1. In this case, in order to obtain (38), we have to use the formula (9).

In general, we have the following,

Theorem 3.1. *If we assume $\omega_{\vec{k}}(t) \in \mathcal{C}^{N+1}$ in T_1, $\omega_{\vec{k}}(t) \in \mathcal{C}^N$ in T_2, but $\omega_{\vec{k}}(t) \notin \mathcal{C}^{N+1}$ in T_2.*

Then, in the semi-classical approximation, we have

$$|b_{\vec{k}}|^2 \sim \frac{\hbar^{2N+2} e^2 \|D^{N+1}\vec{f}\|_{\infty}^2}{\epsilon_{\vec{k}}^{2N+4}}. \tag{39}$$

3.1. Computation of the Number of Produced Pairs

Theorem (3.1) gives a bound of the average number of produced pairs in the $\vec{k}$-state when $t > T_2$. Now, we show a bound of the total average number of produced pairs.

First, we study the following no-physical case. We consider

$$\omega_{\vec{k}}(t) = \begin{cases} \omega_{\vec{k}} & \text{if } |t| > T \\ \bar{\omega}_{\vec{k}} & \text{if } |t| < T \end{cases} \tag{40}$$

In this case

$$|b_{\vec{k}}|^2 = \frac{1}{4} \frac{\left(\epsilon_{\vec{k}}^2 - \bar{\epsilon}_{\vec{k}}^2\right)^2}{\epsilon_{\vec{k}}^2 \bar{\epsilon}_{\vec{k}}^2} \sin^2\left(\frac{2}{\hbar}\bar{\epsilon}_{\vec{k}} T\right) \tag{41}$$

and consequently,

$$N(t > T) \equiv \sum_{\vec{k}\in\mathbb{Z}^3} |b_{\vec{k}}|^2 = +\infty \tag{42}$$

When the hypothesis of Theorem (3.1) are satisfied, then in the semi-classical approximation we have

$$N(t > T_2) \equiv \sum_{\vec{k}\in\mathbb{Z}^3} |b_{\vec{k}}|^2 \sim \frac{\hbar^{2N} \alpha L^3 \|D^{N+1}\vec{f}\|_\infty^2}{(mc^2)^{2N+1} c^2} \tag{43}$$

where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant.

In particular, for $N = 0$ from (38) we can deduce, in the semi-classical approach, that

$$N(t > T_2) = \frac{\alpha E(T_2^-)}{64 mc^2}, \tag{44}$$

where $E(T_2^-)$ is the energy of the field at time T_2.

In general, using the W.K.B. method, we can prove the following:

Theorem 3.2. *If we assume that the electric field is $\mathcal{C}^N(\mathbb{R}\backslash\{T\})$ and $\mathcal{C}^{N-1}$ in T; and we suppose, that the field is switched on and off, then, the average number of produced pairs, when the field is switched off, in the semi-classical approximation, is*

$$\frac{\hbar^{2N} \alpha L^3 \|D^{N+1}\vec{f}\|_\infty^2}{(mc^2)^{2N+1} c^2} \tag{45}$$

In particular, if $\vec{E}(t) \equiv \vec{0}$ when $t > T$, then for $N = 0$, this average number is $\frac{\alpha E(T^-)}{64mc^2}$.

4. THE SCHWINGER FORMULA

In this section, we give a rigorous proof of the Schwinger's formula. In order to obtain this formula we consider the potential vector $\vec{f}(t) = (0, 0, f(t))$ with

$$f(t) = \begin{cases} -cET & \text{if } t < -T \\ cEt & \text{if } -T < t < T \\ cET & \text{if } t > T \end{cases} \tag{46}$$

where we have supposed that $T \gg 1$. In this case, we cannot apply th W.K.B. method, because the function $\omega_{\vec{k}}(t)$ increases to infinity when $T \to \infty$.

First, we will bound the average number of produced pairs in the $\vec{k}$-state at time t. In order to take this bound, we use the diagonalization method in the Schrödinger picture (Haro, 2003). In this picture, the evolution problem is

$$\begin{cases} i\hbar\partial_t \phi = \hat{H}_{\vec{k}}(t)\phi \\ \phi(-T) = \phi_{\vec{k}}^{0,0}(-T) \end{cases} \tag{47}$$

where $\hat{H}_{\vec{k}}(t) = \epsilon_{\vec{k}}(t)(\hat{a}_{\vec{k}}^{+}(t)\hat{a}_{\vec{k}}(t) + \hat{b}_{-\vec{k}}^{+}(t)\hat{b}_{-\vec{k}}(t))$ is the diagonalised Hamiltonian, in the Schrödinger picture, at time t.

A semi-classical solution of the problem (48) is

$$\bar{\phi}_{\vec{k}}(t) = \phi_{\vec{k}}^{0,0}(t) - i\hbar \frac{\dot{\epsilon}_{\vec{k}}(t)}{4\epsilon_{\vec{k}}^2(t)} \phi_{\vec{k}}^{1,1}(t) \tag{48}$$

Now, using Haro (2003), it is easy to verify that

$$\frac{1}{\hbar} \int_{-T}^{T} \|(i\hbar\partial_t - \hat{H}_{\vec{k}}(t))\bar{\phi}_{\vec{k}}(t)\|_2 dt \leq \min \left\{ \frac{2\pi\hbar ceE}{c^2 p_\perp^2 + m^2c^4}, 2\hbar(ceE)^2 \int_{-T}^{T} \frac{1}{\epsilon_{\vec{k}}^3(\tau)} d\tau \right\}$$

where $p_\perp^2 = \frac{4\pi^2\hbar^2}{L^2}(k_1^2 + k_2^2)$. We also have $|\hbar \frac{\dot{\epsilon}_{\vec{k}}(t)}{4\epsilon_{\vec{k}}^2(t)}| \leq \frac{\hbar ceE}{4\epsilon_{\vec{k}}^2(t)}$.

Then, the solution of (48), namely, $\mathcal{T}(t, -T)\phi_{\vec{k}}^{0,0}(-T)$, verifies (Maslov and Fedoriuk, 1981)

$$\begin{aligned} \|\mathcal{T}(t, -T)\phi_{\vec{k}}^{0,0}(-T) - \bar{\phi}_{\vec{k}}(t)\|_2 &\leq \|\phi_{\vec{k}}^{0,0}(-T) - \bar{\phi}_{\vec{k}}(-T)\|_2 \\ &+ \frac{1}{\hbar} \int_{-T}^{T} \|(i\hbar\partial_r - \hat{H}_{\vec{k}}(\tau))\bar{\phi}_{\vec{k}}(t)\|_2 d\tau \\ &\leq \min \left\{ \frac{3\pi\hbar ceE}{c^2 p_\perp^2 + m^2c^4}, \frac{\hbar ceE}{4\epsilon_{\vec{k}}^2(-T)} + 2\hbar(ceE)^2 \int_{-T}^{T} \frac{1}{\epsilon_{\vec{k}}^3(\tau)} d\tau \right\} \sim \mathcal{O}(\hbar) \end{aligned}$$

Now, let $P_{n,\vec{k}}(t)$ be the probability that n pairs are produced in the $\vec{k}$-state at time t, then we can deduce

$$P_{0,\vec{k}}(t) = \left|\langle \phi_{\vec{k}}^{0,0}(t), \mathcal{T}(t,-T)\, \phi_{\vec{k}}^{0,0}(-T)\rangle\right|^2 = 1 + \mathcal{O}(\hbar) > \frac{1}{2}$$

$$P_{1,\vec{k}}(t) = \left|\langle \phi_{\vec{k}}^{1,1}(t), \mathcal{T}(t,-T) \phi_{\vec{k}}^{0,0}(-T)\rangle\right|^2 \leq \min\left\{ \frac{10\pi^2(\hbar ceE)^2}{(c^2 p_\perp^2 + m^2c^4)^2}, \right.$$

$$\left. \times \frac{(\hbar ceE)^2}{16\epsilon_{\vec{k}}^4(t)} + \frac{(\hbar ceE)^2}{16\epsilon_{\vec{k}}^4(-T)} + 4\hbar^2(ceE)^4 \left(\int_{-T}^{T} \frac{1}{\epsilon_{\vec{k}}^3(\tau)} d\tau\right)^2 \right\}$$

Using that (Grib, Mamayev and Mostepanenko, 1994; Marinov and Popov, 1977; Nikishov, 1970)

$$P_{n,\vec{k}}(t) = P_{0,\vec{k}}(t)(1 - P_{0,\vec{k}}(t))^n = P_{0,\vec{k}} \left(\frac{P_{1,\vec{k}}}{P_{0,\vec{k}}}\right)^n$$

we deduce that the average number of produced pairs in the $\vec{k}$-state at time t is

$$N_{\vec{k}}(t) = \sum_{n\in\mathbb{Z}} n P_{n,\vec{k}}(t) = \frac{P_{1,\vec{k}}(t)}{P_{0,\vec{k}}^2(t)}$$

We bound the average number of produced pairs that have the third component of the momentum between eEt_1 and eEt_2. Then, using the previous bounds, we obtain

$$\sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ \frac{2\pi\hbar k_3}{L} \in [eEt_1, eEt_2]}} N_{\vec{k}}(t) \leq 40\pi^3 \frac{L^3}{(2\pi\hbar)^3} \frac{(\hbar ceE)^2}{(mc^2)^2 c^3} ceE(t_2 - t_1) \tag{49}$$

We also calculate the average number of produced pairs that have the third component of the momentum in $(-\infty, -eE(T + \sqrt{\frac{mcT}{eE}})) \cup (eE(T + \sqrt{\frac{mcT}{eE}}), +\infty)$. Then

$$\sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ \frac{2\pi\hbar|k_3|}{L} \geq eE\left(T+\sqrt{\frac{mcT}{eE}}\right)}} N_{\vec{k}}(t) \leq \sum_{\vec{k}\in\mathbb{Z}^3} \frac{(\hbar ceE)^2}{16\epsilon_{\vec{k}}^4(t)} + \sum_{\vec{k}\in\mathbb{Z}^3} \frac{(\hbar ceE)^2}{16\epsilon_{\vec{k}}^4(-T)}$$

$$+ 4\hbar^2(ceE)^4 \sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ \frac{2\pi\hbar|k_3|}{L} \geq eE\left(T+\sqrt{\frac{mcT}{eE}}\right)}} \left(\int_{-T}^{T} \frac{1}{\epsilon_{\vec{k}}^3(\tau)} d\tau\right)^2$$

The first and second terms are bounded by $\frac{\pi^2(\hbar ceE)^2 L^3}{16(2\pi\hbar)^3 c^3 mc^2}$.

In order to bound the third term, first we bound

$$\int_{-T}^{T} \frac{1}{\epsilon_{\vec{k}}^3(\tau)} d\tau \le \frac{1}{\sqrt{c^2 p_\perp^2 + m^2 c^4}} \int_{-T}^{T} \frac{1}{c^2 p_\perp^2 + (ceE\tau)^2 + m^2 c^4} d\tau$$
$$\le \frac{\pi}{eEc(c^2 p_\perp^2 + m^2 c^4)}$$

The third term is less than or equal to

$$(*) \equiv 4\pi \hbar^2 (ceE)^3 \sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ \frac{2\pi\hbar|k_3|}{L} \ge eE\left(T+\sqrt{\frac{mcT}{eE}}\right)}} \frac{1}{c^2 p_\perp^2 + m^2 c^4} \int_{-T}^{T} \frac{1}{\epsilon_{\vec{k}}^3(\tau)} d\tau$$
$$= \frac{4\pi \hbar^2 (ceE)^3 L^3}{(2\pi\hbar)^3} \int_{-T}^{T} \int_{\mathbb{R}^2} \frac{dp_\perp}{c^2 p_\perp^2 + m^2 c^4} \int_{|p_3| \ge eE\left(T+\sqrt{\frac{mcT}{eE}}\right)}$$
$$\times \frac{dp_3}{(c^2 p_\perp^2 + c^2 (p_3 + eE\tau)^2 + m^2 c^4)^{\frac{3}{2}}}$$

Now, using the following bound

$$\int_{|p_3| \ge eE\left(T+\sqrt{\frac{mcT}{eE}}\right)} \frac{dp_3}{(c^2 p_\perp^2 + c^2 (p_3 + eE\tau)^2 + m^2 c^4)^{\frac{3}{2}}} \le \frac{1}{mc^2} \int_{|p_3| \ge eE\left(T+\sqrt{\frac{mcT}{eE}}\right)}$$
$$\times \frac{c|p_3 + eE\tau| dp_3}{(c^2 p_\perp^2 + c^2 (p_3 + eE\tau)^2 + m^2 c^4)^{\frac{3}{2}}} \le \frac{2}{mc^2} \frac{1}{(c^2 p_\perp^2 + eEmc^3 T + m^2 c^4)^{\frac{1}{2}}}$$

we obtain

$$(*) \le \frac{32\pi^2 \hbar^2 (eEc)^3 L^3}{c^3 mc^2 (2\pi\hbar)^3} \frac{T}{(mc^3 eET)^{\frac{1}{4}} (mc^2)^{\frac{1}{2}}}$$

Consequently,

$$\sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ \frac{2\pi\hbar|k_3|}{L} \ge eE\left(T+\sqrt{\frac{mcT}{eE}}\right)}} N_{\vec{k}}(t) \le \frac{\pi^2 (\hbar ceE)^2 L^3}{8(2\pi\hbar)^3 c^3 mc^2}$$
$$+ \frac{32\pi^2 \hbar^2 (eEc)^3 L^3}{c^3 mc^2 (2\pi\hbar)^3} \frac{T}{(mc^3 eET)^{\frac{1}{4}} (mc^2)^{\frac{1}{2}}} \qquad (50)$$

Remark 4.1. These results will be very important in order to compute the average number of produced pairs.

Now we study the problem (Bagrov, Gitman and Shvartsman, 1975; Haro, 2003)

$$\ddot{u}_{\vec{k}} + \frac{1}{\hbar^2}\left(c^2 p_\perp^2 + c^2\left(\frac{2\pi\hbar k_3}{L} + eEt\right)^2 + m^2c^4\right)u_{\vec{k}} = 0; \quad t \in (-T, T) \tag{51}$$

If we make the change $y = \sqrt{\frac{2c}{\hbar eE}}(p_3 + eEt)$, then we obtain the differential equation

$$u''_{\vec{k}} + \left(\frac{1}{4}y^2 - A\right)u_{\vec{k}} = 0 \tag{52}$$

where $A = \frac{-1}{2eEc\,\hbar}(c^2 p_\perp^2 + m^2c^4)$.

One independent set of solutions of (52) is

$$u_{1,\vec{k}}(y) = \exp\left(-\frac{i}{4}y^2\right) M\left(-\frac{i}{2}A + \frac{1}{4}, \frac{1}{2}, \frac{i}{2}y^2\right) \tag{53}$$

$$u_{2,\vec{k}}(y) = \frac{1}{\sqrt{2}}\exp\left(-\frac{i}{4}y^2\right) y \exp\left(-\frac{i\pi}{4}\right) M\left(-\frac{i}{2}A + \frac{3}{4}, \frac{3}{2}, \frac{i}{2}y^2\right) \tag{54}$$

where M is the Kummer's function.

We now define

$$\varphi_{\vec{k}}^{+}(y) = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{4} + \frac{i}{2}A\right)}u_1(y) + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{i}{2}A\right)}u_2(y) \tag{55}$$

$$\varphi_{\vec{k}}^{-}(y) = -i\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{4} - \frac{i}{2}A\right)}u_1(y) + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{i}{2}A\right)}u_2(y) \tag{56}$$

Then for $y < 0$, we have (Abramowitz and Stegun, 1968)

$$\varphi_{\vec{k}}^{+}(y) = \bar{A}B\exp\left(\frac{\pi A}{4}\right)\exp\left(-\frac{i\pi}{8}\right)\left(\frac{y^2}{2}\right)^{-\frac{1}{4}-\frac{i}{2}A}\exp\left(\frac{i}{4}y^2\right)[1 + R(A, y^2)] \tag{57}$$

$$\varphi_{\vec{k}}^{-}(y) = -\bar{A}B\exp\left(\frac{\pi A}{4}\right)\exp\left(\frac{i\pi}{8}\right)\left(\frac{y^2}{2}\right)^{-\frac{1}{4}+\frac{i}{2}A}\exp\left(-\frac{i}{4}y^2\right)[1 + R(A, y^2)], \tag{58}$$

with

$$\bar{A} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{i}{2}A\right)\Gamma\left(\frac{3}{4} + \frac{i}{2}A\right)}; \quad B = \frac{\Gamma\left(\frac{1}{4} + \frac{i}{2}A\right)}{\Gamma\left(\frac{1}{4} - \frac{i}{2}A\right)} + i\frac{\Gamma\left(\frac{3}{4} + \frac{i}{2}A\right)}{\Gamma\left(\frac{3}{4} - \frac{i}{2}A\right)}$$

In order to obtain the bound of the function $R(A, y^2)$ we have used Nikiforov and Ouvarov (1976), and we have obtained

$$|R(A, y^2)| \leq -K\frac{A}{y^2}\exp\left(-\frac{A\pi}{2}\right)$$

where K is a positive a-dimensional constant independent of A and y.

For $y > 0$, we have

$$\varphi^+_{\vec{k}}(y) = \exp\left(\frac{\pi A}{4}\right)\left\{2\bar{A}\exp\left(\frac{i\pi}{8}\right)\left(\frac{y^2}{2}\right)^{-\frac{1}{4}+\frac{i}{2}A}\exp\left(\frac{-iy^2}{4}\right)[1 + R(A, y^2)]\right.$$
$$\left. + \ \bar{A}C\exp\left(\frac{-i\pi}{8}\right)\left(\frac{y^2}{2}\right)^{-\frac{1}{4}-\frac{i}{2}A}\exp\left(\frac{iy^2}{4}\right)[1 + R(A, y^2)]\right\} \tag{59}$$

$$\varphi^-_{\vec{k}}(y) = \exp\left(\frac{\pi A}{4}\right)\left\{2i\bar{A}^*\exp\left(\frac{-i\pi}{8}\right)\left(\frac{y^2}{2}\right)^{-\frac{1}{4}-\frac{i}{2}A}\exp\left(\frac{iy^2}{4}\right)[1 + R(A, y^2)]\right.$$
$$\left. + \ \bar{A}C\exp\left(\frac{i\pi}{8}\right)\left(\frac{y^2}{2}\right)^{-\frac{1}{4}+\frac{i}{2}A}\exp\left(\frac{-iy^2}{4}\right)[1 + R(A, y^2)]\right\} \tag{60}$$

with

$$C = \frac{\Gamma\left(\frac{1}{4} + \frac{i}{2}A\right)}{\Gamma\left(\frac{1}{4} - \frac{i}{2}A\right)} - i\frac{\Gamma\left(\frac{3}{4} + \frac{i}{2}A\right)}{\Gamma\left(\frac{3}{4} - \frac{i}{2}A\right)}$$

Remark 4.2. For the derivate we obtain similar expressions to (57)–(60).

Now, we study the case $\frac{2\pi\hbar|k_3|}{eEL} \leq T - \sqrt{\frac{Tmc}{eE}}$. Since $y = \sqrt{\frac{2c}{\hbar eE}}(p_3 + eEt)$ we have $y(-T) < 0$ and $y(T) > 0$. Therefore, from the formula (9) and the expressions (57)–(60) it is easy to see that for $\frac{2\pi\hbar|k_3|}{eEL} \leq T - \sqrt{\frac{Tmc}{eE}}$, we have

$$|b_{\vec{k}}|^2 = \frac{|C|^2}{|B|^2}(1 + G(\vec{p}, T)) = \exp\left(-\frac{\pi}{eEc\hbar}(c^2p_\perp^2 + m^2c^4)\right) + F(\vec{p}, T), \tag{61}$$

with

$$|F(\vec{p}, T)| \leq \tilde{K}\frac{c^2p_\perp^2 + m^2c^4}{mc^3TeE}\exp\left(-\frac{3\pi}{4eEc\hbar}(c^2p_\perp^2 + m^2c^4)\right)$$

where $\tilde{K}$ is an a-dimensional constant independent of T, $p_\perp$, and $\hbar$.

With these results we can calculate the average number of produced pairs per unit time and unit volume, when $T \to \infty$.

$$\sum_{\vec{k}\in\mathbb{Z}^3} \frac{|b_{\vec{k}}|^2}{2TL^3} = \sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ \frac{2\pi\hbar|k_3|}{L} \geq eE\left(T+\sqrt{\frac{Tmc}{eE}}\right)}} \frac{|b_{\vec{k}}|^2}{2TL^3} + \sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ \frac{2\pi\hbar|k_3|}{L} \leq eE\left(T-\sqrt{\frac{Tmc}{eE}}\right)}} \frac{|b_{\vec{k}}|^2}{2TL^3}$$

$$+ \sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ eE\left(T-\sqrt{\frac{Tmc}{eE}}\right) \leq \frac{2\pi\hbar|k_3|}{L} \leq eE\left(T+\sqrt{\frac{Tmc}{eE}}\right)}} \frac{|b_{\vec{k}}|^2}{2TL^3}$$

Then, due to the expressions (49) and (50), we have

$$\lim_{T\to\infty} \sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ eE\left(T-\sqrt{\frac{Tmc}{eE}}\right) \leq \frac{2\pi\hbar|k_3|}{L} \leq eE\left(T+\sqrt{\frac{Tmc}{eE}}\right)}} \frac{|b_{\vec{k}}|^2}{2TL^3} \leq \lim_{T\to\infty} 40\pi^3$$

$$\times \frac{1}{(2\pi\hbar)^3} \frac{(\hbar ceE^2)}{(mc^2)^2c^3} \sqrt{\frac{eEmc^3}{T}} = 0$$

$$\lim_{T\to\infty} \sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ \frac{2\pi\hbar|k_3|}{L} \geq eE(T+\sqrt{T})}} \frac{|b_{\vec{k}}|^2}{2TL^3} \leq \lim_{T\to\infty}$$

$$\times \left[\frac{\pi^2(\hbar ceE)^2}{16T(2\pi\hbar)^3c^3mc^2} + \frac{16\pi^2\hbar^2(eEc)^3}{c^3mc^2(2\pi\hbar)^3} \frac{1}{(mc^3eET)^{\frac{1}{4}}(mc^2)^{\frac{1}{2}}} \right] = 0$$

And, due to the formula (61), we obtain

$$\lim_{T\to\infty} \sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ \frac{2\pi\hbar|k_3|}{L} \leq eE\left(T-\sqrt{\frac{Tmc}{eE}}\right)}} \frac{|b_{\vec{k}}|^2}{2TL^3} = \lim_{T\to\infty} \frac{2\left(T - \sqrt{\frac{Tmc}{eE}}\right)}{2T(2\pi\hbar)^3}$$

$$\int_{\mathbb{R}^2} \exp\left(-\frac{\pi(c^2p_\perp^2 + m^2c^4)}{\hbar ceE}\right) dp_\perp = \frac{E^2\alpha}{8\pi^3\hbar} \exp\left(-\frac{\pi m^2c^4}{\hbar ceE}\right)$$

Consequently, we have proved that

$$\lim_{T\to\infty} \sum_{\vec{k}\in\mathbb{Z}^3} \frac{|b_{\vec{k}}|^2}{2TL^3} = \frac{E^2\alpha}{8\pi^3\hbar} \exp\left(-\frac{\pi m^2c^4}{\hbar ceE}\right) \tag{63}$$

Remark 4.3. In Holstein (1999), and Nikishov (1970), the authors calculate the quantity

$$\frac{1}{2TL^3} \sum_{\vec{k}\in\mathbb{Z}^3} \lim_{T\to\infty} |b_{\vec{k}}|^2 = \frac{1}{2T(2\pi\hbar)^3} \int_{\mathbb{R}^3} \exp\left(-\frac{\pi(c^2p_\perp^2 + m^2c^4)}{\hbar ceE}\right) d\vec{p}$$

and make the replacement $\int_{\mathbb{R}} dp_3 \to 2eET$ in order to obtain the formula (63). Clearly, this argument is meaningless.

Using the same argument, we can prove that, when $T \to \infty$, the relative probability that a pair is produced per unit time and unit volume, is (Haro, in press)

$$\lim_{T\to\infty} \sum_{\vec{k}\in\mathbb{Z}^3} \frac{1}{2TL^3} \frac{|b_{\vec{k}}|^2}{|a_{\vec{k}}|^2} = \frac{E^2\alpha}{8\pi^3\hbar} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \exp\left(-\frac{n\pi m^2 c^4}{\hbar c e E}\right) \tag{64}$$

in contrast with the interpretation given by Schwinger and other authors (Greiner, Müller and Rafelski, 1985; Itzykson and Zuber, 1980; Popov, 1972; Schwinger, 1951).

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